Funciones elementales

(A) $f(z) = e^x \cos y + i e^x \operatorname{sen} y \qquad \operatorname{Dom} f = \mathbb{C}$

- $f(0) = e^0 \cos 0 + e^0 \operatorname{sen} 0 = 1$

- Holomorfa? $\left.\begin{array}{l} u(x,y) = e^x \cos y \\ v(x,y) = e^x \operatorname{sen} y \end{array}\right\}$ son C^1

C-R? $u'_x(x,y) = e^x \cos y \qquad v'_y(x,y) = e^x \cos y$

$u'_y(x,y) = -e^x \operatorname{sen} y \qquad v'_x(x,y) = e^x \operatorname{sen} y$

Se verifica C-R en todo $\mathbb{C}$.

$f'(z) = u'_x(x,y) + i v'_x(x,y)$

f es Holomorfa en $\mathbb{C}$ (entera)

$= e^x \cos y + i e^x \operatorname{sen} y$

$f'(z) = f(z)$

Observe: $\left\{\begin{array}{l} f'(z) = f(z) \\ f(0) = 1 \end{array}\right.$

$$\boxed{f(z) = e^z}$$

- $f(z + 2k\pi i) = f(z) \rightarrow$ periódica, período: $2\pi i$

- si $z = x \in \mathbb{R}$, $f(z) = f(x) = e^x$

$$- e^{\overset{x_1+iy_1}{z_1}} \cdot e^{\overset{x_2+iy_2}{z_2}} = \left[e^{x_1}\cos y_1 + ie^{x_1}\operatorname{sen} y_1\right]\cdot\left[e^{x_2}\cos y_2 + ie^{x_2}\operatorname{sen} y_2\right]$$

$$= e^{x_1}\cdot e^{x_2}\left[\cos y_1 + i\operatorname{sen} y_1\right]\left[\cos y_2 + i \operatorname{sen} y_2\right]$$

$$= e^{x_1}\cdot e^{x_2}\left[\underbrace{\cos y_1\cos y_2 - \operatorname{sen} y_1 \operatorname{sen} y_2}_{\cos(y_1+y_2)} + i\underbrace{\left(\operatorname{sen} y_1 \cos y_2 + \operatorname{sen} y_2 \cos y_1\right)}_{\operatorname{sen}(y_1+y_2)}\right]$$

$$= e^{x_1+x_2}\left[\cos(y_1+y_2) + i\operatorname{sen}(y_1+y_2)\right] = e^{z_1+z_2}$$

$$\boxed{e^{z_1}\cdot e^{z_2} = e^{z_1+z_2}}$$

$\to e^{nz_1} = e^{z_1+z_1+\cdots+z_1} = (e^{z_1})^n \quad n\in\mathbb{N}$

$\to e^{z_1-z_1} = e^0 = 1 = e^{z_1}\cdot e^{-z_1} \Rightarrow (e^{z_1})^{-1} = e^{-z_1} = \dfrac{1}{e^{z_1}}$

$\to e^{z_1-z_2} = e^{z_1}\cdot e^{-z_2} = \dfrac{e^{z_1}}{e^{z_2}}$

$\to e^{mz} = (e^z)^m \ , \ m\in\mathbb{Z}$

$- |e^z| = |e^{x+iy}| = |e^x\cos y + ie^x \operatorname{sen} y| = e^x > 0 \quad (\Rightarrow e^z \neq 0)$

$\arg(e^z) = y + 2k\pi, \quad k\in\mathbb{Z}$

$- \quad w = e^z \to$ dado w, quien es z? $\quad (w\neq 0)$

$\rho e^{i\theta} = e^x\cdot e^{iy} \Rightarrow \begin{cases}\rho = e^x\\ \theta = y+2k\pi\end{cases} \Rightarrow \begin{cases} x = \ln\rho \\ y = \theta + 2k\pi \end{cases} \quad k\in\mathbb{Z}$

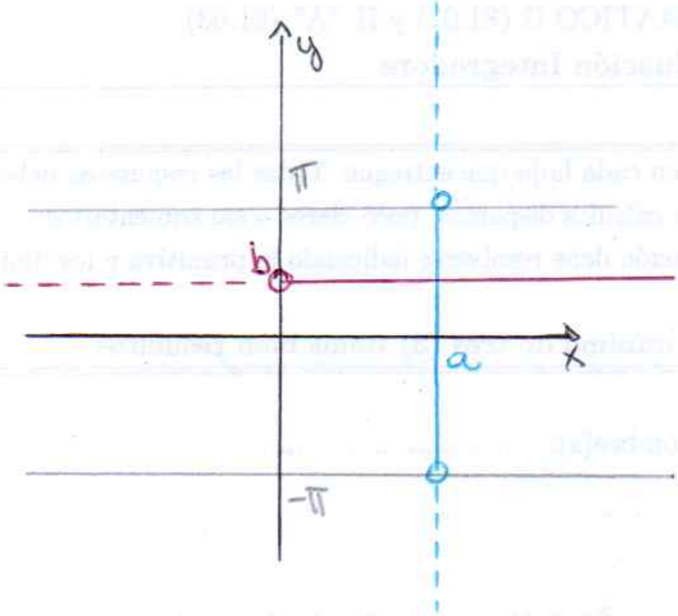
$z = \ln\rho + i(\theta+2k\pi) \ , \ k\in\mathbb{Z}$

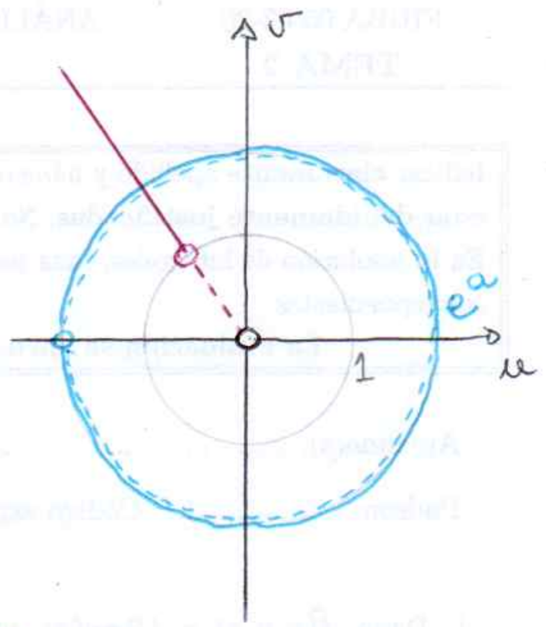
↳ infinitos z

Ej: $e^z = -1 = 1e^{i\pi} \Rightarrow \begin{cases} 1 = e^x \\ \pi = y+2k\pi\end{cases} \Rightarrow \begin{matrix} x = \ln 1 = 0 \\ y = \pi + 2k\pi, \ k\in\mathbb{Z}\end{matrix}$

$$\boxed{z = i(\pi+2k\pi)}$$

VISUALIZACION DE EXPONENCIAL





$$z = a+iy \longrightarrow e^z = e^{a+iy} = e^a(\cos y + i\,\mathrm{sen}\, y)$$

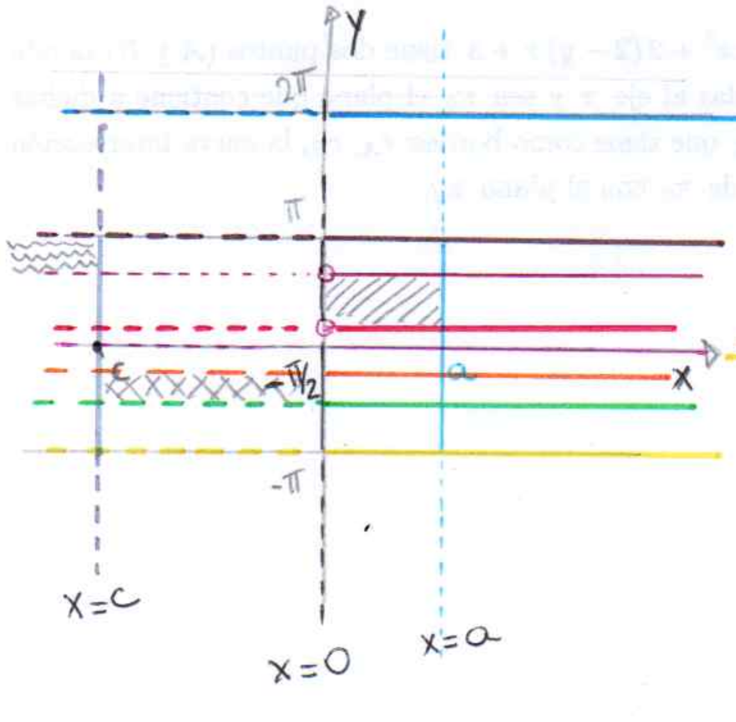
$a>0$

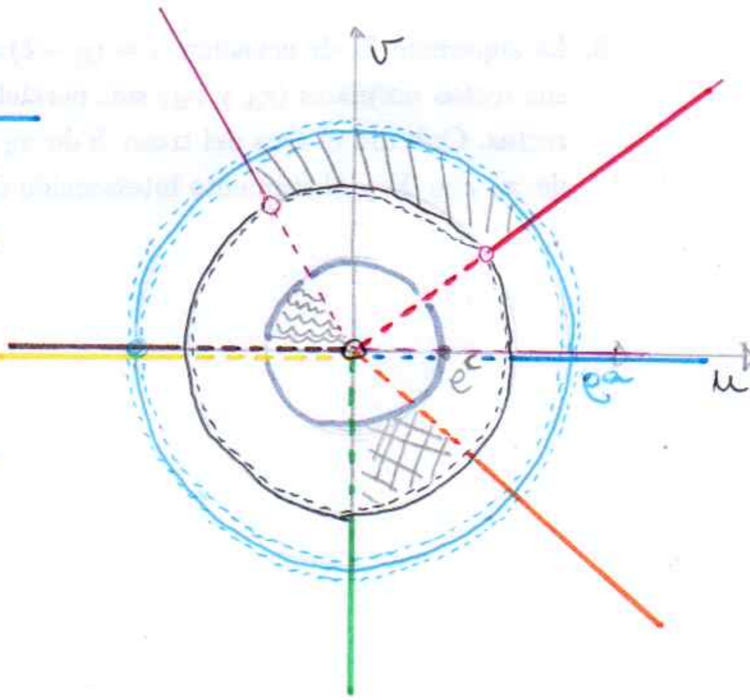
$$\left.\begin{array}{l} u = e^a \cos y \\ v = e^a\,\mathrm{sen}\, y \end{array}\right\} \begin{array}{l} u^2+v^2 = e^{2a} \\ \sqrt{u^2+v^2} = e^a > 1 \end{array}$$

$$z = x+ib \longrightarrow e^z = e^{x+ib} = e^x(\cos b + i\,\mathrm{sen}\, b)$$

$0 \le b \le \pi$

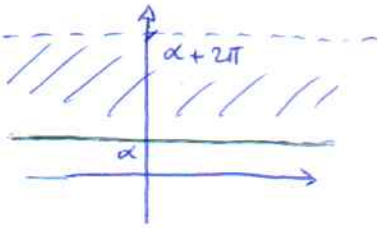
$$\left.\begin{array}{l} u = e^x \cos b \\ v = e^x\,\mathrm{sen}\, b \end{array}\right\} \begin{array}{l} \cdot\ \frac{v}{u} = \mathrm{tg}\, b \quad \text{si } b \neq \pi/2 \\ \cdot\ u = 0,\ v>0 \quad \text{si } b = \pi/2 \end{array}$$



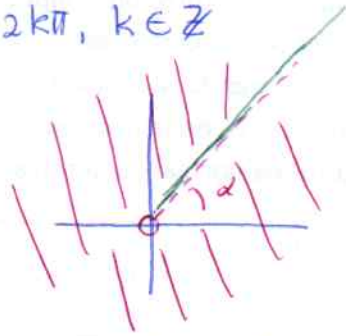


(B) INVERSA DE EXPONENCIAL.

$$\omega = e^{z} \Rightarrow |\omega| = e^{x}$$
$$\arg(\omega) = y + 2k\pi,\ k \in \mathbb{Z}$$



biyectiva



$$\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\} \to \{\omega \in \mathbb{C} : \omega \neq 0\}$$

Si $\omega = \rho e^{i\theta} \Rightarrow \rho = e^{x}$, $\theta = y + 2k\pi$ $\Rightarrow$ $x = \ln \rho = \ln|\omega|$, $y = \arg(\omega)$

Función logaritmo:

$$z = \log(\omega)$$
$$x + iy = \ln|\omega| + i \arg(\omega)$$

↳ para que sea función, especificar un valor de $\arg(\omega)$.

Ojo: cambio nombre de variables para que "z" sea la indep.

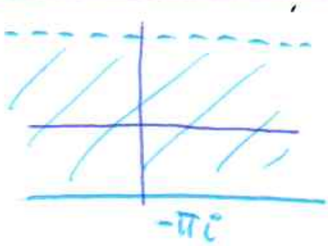
$$\boxed{\omega = \log(z) = \ln|z| + i\arg(z)}, \quad \alpha \leq \arg(z) < \alpha + 2\pi$$

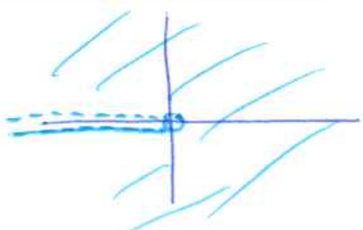
Usualmente:

Logaritmo principal: $\omega = \operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$

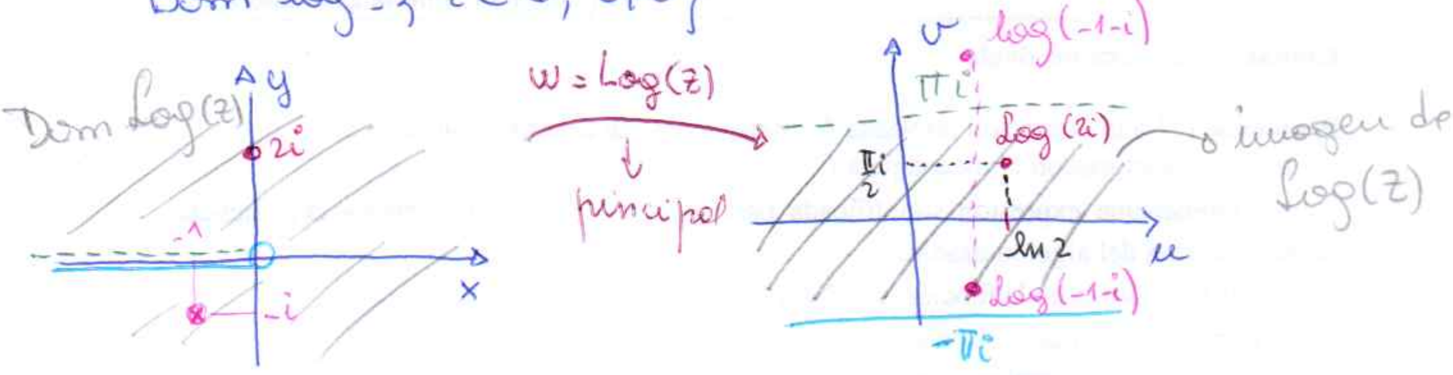
o:

$$\omega = \log(z) = \ln|z| + i\arg(z), \quad 0 \leq \arg(z) < 2\pi$$





$\text{Dom } \log = \{ z \in \mathbb{C},\ z \neq 0 \}$



$u = \ln|z|$

$v = \text{Arg}(z)$

<u>Ejemplo</u>: $\text{Log}(1) = \ln|1| + i\,\text{Arg}(1) = 0 + i0 = 0$

$\text{Log}(2i) = \ln|2i| + i\,\text{Arg}(2i) = \ln 2 + i\,\pi/2$

$\text{Log}(-1-i) = \ln|-1-i| + i\,\text{Arg}(-1-i) = \ln\sqrt{2} - \frac{3\pi}{4} i$

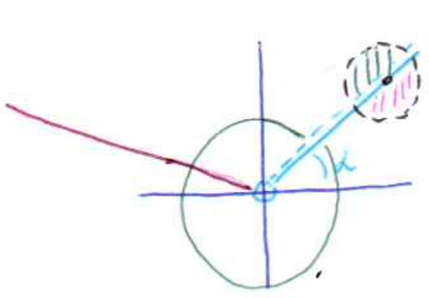
Si tomamos $\log(z)$ con $0 \leq \arg(z) < 2\pi$

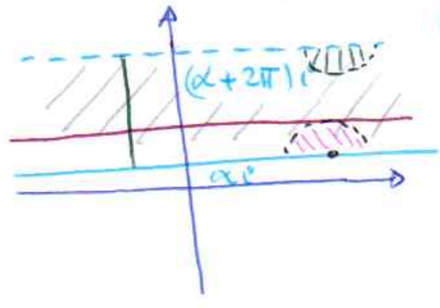
$\log(-1-i) = \ln|-1-i| + i\,\arg(-1-i) = \ln\sqrt{2} + \underbrace{\frac{5\pi}{4} i}_{-\frac{3}{4}\pi i + 2\pi i}$

Consideremos: $\log(z) = \ln|z| + i\arg(z)$, $\alpha \leq \arg z < \alpha + 2\pi$.

- ES continua en $\mathbb{C} - \{ \text{semirrecta de } z : \arg(z) = \alpha \}$

$\mathbb{C} - \underbrace{\{ z : \arg z = \alpha \}}_{\text{"corte de rama"}}$ → $z = 0$: punto de ramificación





• Derivable? $f(z) = u(r,\theta) + i\,v(r,\theta) = \ln r + i\theta$

$$\left.\begin{array}{l} u(r,\theta) = \ln r \\ v(r,\theta) = \theta \end{array}\right\} \text{ dif para } r>0,\ \alpha<\theta<\alpha+2\pi$$

C-R en polares (prob 23): $\begin{array}{l} r u'_r = v'_\theta \\ r v'_r = -u'_\theta \end{array}$

$$\left.\begin{array}{ll} u'_r = \frac{1}{r} & v'_\theta = 1 \\ v'_r = 0 & u'_\theta = 0 \end{array}\right\} \Rightarrow \text{se verifica C.R para } r>0,\ \alpha<\theta<\alpha+2\pi$$

Derivada: $f'(z) = e^{-i\theta}\,(u'_r(r,\theta) + i\,v'_r(r,\theta))$

$$= e^{-i\theta}\cdot\frac{1}{r} + i0 = \frac{1}{e^{i\theta} r} = \frac{1}{z}$$

$\boxed{f'(z) = \frac{1}{z}}$ $\rightarrow$ para $z\neq 0$, $\arg(z)\neq\alpha$

– $e^{\log z} = e^{\ln|z| + i\arg(z)} = e^{\ln|z|}\cdot e^{i\arg(z)} = |z|\left(\cos(\arg(z)) + i\,\mathrm{sen}(\arg(z\right.$

$\boxed{e^{\log z} = z}$ ✓

$\log(e^z) = \log(e^{x+iy}) = \ln|e^{x+iy}| + i\arg(e^{x+iy}) =$

$= \ln e^x + i(y + 2k\pi) \quad k\in\mathbb{Z}$

$\boxed{\log(e^z) = z + 2k\pi i \quad k\in\mathbb{Z}.}$

Ⓒ Exponentes complejos

$$f(z) = z^c = \underbrace{e^{c \cdot \log(z)}}_{} \qquad c \in \mathbb{C}$$

para que sea "función", tomar una rama de log.

$$z^c = e^{c \cdot \log(z)} = e^{c(\ln|z| + i \arg(z))} = e^{c \ln|z|} \cdot e^{i\, c \cdot \arg(z)}$$

$$= |z|^c \cdot e^{i(\theta + 2k\pi) \cdot c}$$

- Continua en $\mathbb{C}$ excepto en el corte de rama
- Derivable en $\mathbb{C}$ excepto en el corte de rama

$$f'(z) = e^{c \cdot \log(z)} \cdot c \cdot \frac{1}{z} = c \cdot \frac{z^c}{z} = c \cdot z^{c-1}$$

- Observar: • si $c = n \in \mathbb{N}$:

$$z^c = z^n = e^{n \log(z)} = e^{n \ln|z|} \cdot e^{i n \arg(z)} = |z|^n \cdot e^{i n(\theta + 2k\pi)} = |z|^n e^{i n \theta}$$

• si $c = \frac{1}{n}$, $n \in \mathbb{N}$:

$$z^c = z^{1/n} = e^{\frac{1}{n} \log(z)} = e^{\frac{1}{n} \ln|z|} \cdot e^{i \frac{1}{n} \arg(z)} = |z|^{1/n} \cdot e^{i \frac{1}{n}(\theta + 2k\pi)}$$

$$\Rightarrow \quad z^{1/n} = |z|^{1/n} \cdot e^{i \frac{\theta}{n}} \cdot e^{i \frac{2k\pi}{n}}$$

Ⓓ TRIGONOMÉTRICAS

$$\operatorname{sen} z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

TO BE CONTINUED ...